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HEAVY TRANSVERSALS AND INDECOMPOSABLE HYPERGRAPHS

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A weighted hypergraph is a hypergraph $H=(V,E)$ with a weighting function $w:V\to\mathcal{R}$, where $\mathcal{R}$ is the set of reals. A multiset $S\subseteq V$ generates a partial hypergraph H^S with edges $\{e\in E : |e\cap S|>w(S)\}$, where both the cardinality $|e\cap S|$ and the total weight $w(S)$ are counted with multiplicities of vertices in S. The transversal number of H is represented by $\tau(H)$. We prove the following: there exists a function $f(n)$ such that, for any weighted n-Helly hypergraph H, $\tau(H^B)\le 1$, for all multisets $B\subseteq V$ if and only if $\tau(H^A)\le 1$, for all multisets $A\subseteq V$ with $|A|\le f(n)$. We provide lower and upper bounds for $f(n)$ using a link between indecomposable hypergraphs and critical weighted n-Helly hypergraphs.

1. Introduction

The notion of consensus has many applications and so is an important tool in several academic disciplines. In biology, for example, finding consensus patterns among several DNA sequences is an important step in the comparison of those sequences [7,8]. We focus on the problem of determining when a consensus rule, based on some notion of counting, produces a consensus. Roughly this problem can be formulated in the following way. Consider a finite set of candidates in an election, and a given family of properties. Each property represents a group of candidates with a given characteristic, and so each property is a subset of the candidates. Each voter chooses one candidate that embodies several properties. Let us say that a property is important if

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sufficiently many of the voters choose a candidate with this property. A candidate that has all of the important properties is a *consensus*. The problem is to determine conditions that guarantee a consensus candidate. Observe that a consensus candidate need not be among the candidates that receive a vote. Also observe that a candidate may receive more than one vote. The hypergraph model we adopt reflects these properties of the consensus problem.

The underlying structure of this problem is a hypergraph $H=(V,E)$, where the vertices represent the candidates in the election, and the edge set represents the given properties of the candidates. A "counting rule" determines a weighting of the vertices. Specifically, a *weighted hypergraph* is a hypergraph $H=(V,E)$ with a weighting function $w:V\to\mathcal{R}$, where $\mathcal{R}$ is the set of reals. A multiset S of vertices generates a partial hypergraph H^S with edges $\{e\in E:|e\cap S|>\sum_{x\in S} m_S(x)w(x)\}$, where $m_S(x)$ is the mutiplicity of x in S. The multiset S is called a heavy transversal of H^S. The edges of the hypergraph H^S are the important properties with respect to S which represents the outcome of an election. The transversal number of H, denoted by $\tau(H)$, is the minimum number of vertices of H intersecting all edges of H. Observe that if $\tau(H^S)=1$, then there is a consensus candidate with respect to S. The main result of this paper is the following: there exists a function $f(n)$ such that, for any weighted n-Helly hypergraph H, $\tau(H^B)\le 1$, for all $B\subseteq V$ if and only if $\tau(H^A)\le 1$, for all $A\subseteq V$ with $|A|\le f(n)$. This has the following interpretation: given the candidates, properties, and the "counting rule" determined by the weighting function, every election has a consensus if and only if every election involving at most $f(n)$ voters has a consensus. Using results on indecomposable hypergraphs we prove $\binom{n}{3}2^{\frac{n-6}{2}}\le f(n)\le n(n+1)^{\frac{n+1}{2}}$. However, as we point out later, determining bounds on $f(n)$ is not equivalent to the problem of determining bounds on indecomposable hypergraphs.

2. Heavy transversals

A collection of elements – possibly with some elements repeated – is a *multiset*. A hypergraph H is a pair $H=(V(H),E(H))$, where $V=V(H)$ is a finite set, the set of vertices, and $E=E(H)$ is a multiset of nonempty subsets of V, the multiset of edges. Because E is a multiset, an edge e may appear more than once in E; the multiplicity of e in E is denoted $m_E(e)$. In general, the multiplicity of the element b in the multiset B is denoted $m_B(b)$, and we define $m_B(x)=0$, for $x\notin B$. The cardinality of a multiset is the number of elements with multiplicity; so, $|E| = \sum_{e\in E} m_E(e)$. An edge appearing more than once is a *multiple edge*. If H has no multiple edges, then it is a *simple* hypergraph. When we wish to emphasize that a hypergraph H may contain multiple edges, we shall refer to H as a *multihypergraph*.

Let $H=(V,E)$ be a multihypergraph. For $F\subseteq E$, the hypergraph H' with $E(H')=F$ and $V(H')=\bigcup_{f\in F} f$ is called the *partial hypergraph* of H induced by F. If $V(H')=V(H)$ then H' is a *spanning* partial hypergraph of H. For $X\subseteq V$, the hypergraph H_X with $V(H_X)=X$ and $E(H_X)=\{e\cap X : e\in E, e\cap X\neq\emptyset\}$ is the *subhypergraph* of H generated by X. For $T\subseteq V$, let H^T be the partial hypergraph of H generated by $\{e\in E : |e\cap T|>0\}$.

A set $T\subseteq V(H)$ is a *transversal* of H if $H^T=H$. The *transversal number* $\tau(H)$ is the minimum cardinality of a transversal of H. A hypergraph is k*-intersecting* if every k edges of H have a common vertex. The *intersection number* $\kappa(H)$ is the maximum integer k such that H is k-intersecting. Hypergraph H has the n*-Helly property* (we also say that H is n-Helly) if every partial hypergraph $H'\subseteq H$ with $\kappa(H')\geq n$ satisfies $\tau(H')\leq 1$. The *Helly-number* of H is the minimum n such that H is n-Helly. Berge and Duchet [3] characterized hypergraphs with the n-Helly property as follows.

Theorem [Gilmore-criterion]. *A hypergraph H is n-Helly if and only if for every $A\subseteq V(H)$, $|A|=n+1$, the partial hypergraph H' with edge set $\{e\in E(H) : |e\cap A|\geq n\}$ satisfies $\tau(H')\leq 1$.*

A hypergraph H is τ*-critical* if the removal of any hyperedge reduces the transversal number of H. Note that a τ-critical hypergraph is simple. If H is τ-critical such that $\tau(H)=2$ and $E(H)=\{e_1,\dots,e_{k+1}\}$, then $\kappa(H)=k\geq 1$, $\bigcap_{i=1}^{k+1} e_i=\emptyset$, and for every $1\leq j\leq k+1$, there is a vertex $a_j\in\bigcap_{i\neq j} e_i$. Such a set $\{a_1,\dots,a_{k+1}\}$ is called a *kernel* of H. Using the notion of a kernel, the Gilmore-criterion has the following obvious rephrasing: a hypergraph H is n-Helly if and only if every τ-critical partial hypergraph of H with $\tau=2$ has a kernel of cardinality at most n. Our main result, Theorem 2.2, extends the Gilmore-criterion to weighted hypergraphs.

We call $(H;w)=(V,E;w)$ a *weighted* hypergraph if vertices of H are assigned real number weights by some weighting function $w: V\to\mathcal{R}$. For a multiset of vertices A, we denote by H^A the (weighted) partial hypergraph of H generated by $\{e\in E : |e\cap A|>w(A)\}$, where $|e\cap A|=\sum_{v\in e} m_A(v)$ and $w(A)=\sum_{v\in A} m_A(v)w(v)$. The multiset A of vertices is called a *heavy* *transversal* of H if $H^A = H$. In case of $w \equiv 0$, heavy transversals of H are its transversals.

Define a partial order $\subseteq_m$ on multisets of V in the following natural way: $B \subseteq_m A$ if and only if, for all $v \in V$, $m_B(v) \le m_A(v)$. A multiset of vertices A in a weighted multihypergraph $(H;w)$ is *critical* if $\tau(H^A) > 1$ and, $\tau(H^B) \le 1$, for all proper multisets $B \subseteq_m A$. Observe that a set of vertices A is critical if and only if $\tau(H^A) > 1$ and, $\tau(H^B) \le 1$, for all proper subsets $B \subset A$. Define

$$\sigma_m(H;w) = \max\{|A| : A \text{ is a critical multiset of vertices in } (H;w)\},$$

and similarly define the non-multiset version

$$\sigma_s(H;w) = \max\{|A| : A \text{ is a critical set of vertices in } (H;w)\}.$$

Clearly, $\sigma_m(H;w) \ge \sigma_s(H;w)$. Let $\mathcal{H}_n$ denote the family of all finite n-Helly hypergraphs. Set

$$f_m(n) = \max\{\sigma_m(H;w) : \ (H;w) \text{ with } H \in \mathcal{H}_n\},$$

and

$$f_s(n) = \max\{\sigma_s(H;w) : \ (H;w) \text{ with } H \in \mathcal{H}_n\}.$$

A priori the finiteness of these functions is not clear. Clearly $f_m(n) \ge f_s(n)$. Our first observation (Lemma 2.1) is technical and implies that it suffices to consider just one function $f(n) = f_s(n) = f_m(n)$, even in the presence of a restriction on the range of the weighting function. The *range* of the weighting function $w: V \to \mathcal{R}$ is denoted $\mathcal{R}(w)$.

Lemma 2.1. *Suppose $(H;w)$ is a weighted n-Helly multihypergraph. For any critical <u>multiset</u> of vertices A in $(H;w)$, there exists a weighted n-Helly multihypergraph $(H^*;w^*)$ such that $\mathcal{R}(w) \supseteq \mathcal{R}(w^*)$ and such that $(H^*;w^*)$ contains a critical <u>set</u> of vertices A^* satisfying $|A^*| = |A|$.*

Proof. Assume that $\tau(H^A) \ge 2$, for some critical multiset of vertices A in the weighted n-Helly hypergraph $(H;w)$. Let $k = \kappa(H^A)$, and consider the partial hypergraph K defined by $k+1$ hyperedges $e_1, \ldots, e_{k+1} \in E(H^A)$ with no vertex in common. Observe that $k < n$, because otherwise, $\tau(K) \le 1$, by the n-Helly property of H. Also, $k = \kappa(H^A)$ implies that $\bigcap_{i \ne j} e_i \ne \emptyset$, for all $j = 1, \ldots, k+1$. Set $U = \bigcup_{i=1}^{k+1} e_i$.

Define the set A^* as the set obtained from the multiset A by replacing each $a \in A$ by $m_A(a)$ distinct copies of a. Set $U^* = (U \setminus A) \cup A^*$. Similarly define, for every $i = 1, \ldots, k+1$, the set e_i^* as the set obtained from the set e_i

by replacing each $a \in A \cap e_i$ with all $m_A(a)$ distinct copies of a. Consider the weighted hypergraph H^* with vertices U^*, hyperedges $e_1^*, e_2^*, \ldots, e_{k+1}^*$, and weighting function inherited from w.

Observe that $e_1^*, \ldots, e_{k+1}^*$ have no point in common, but $\bigcap_{i \neq j} e_i^* \neq \emptyset$, for all $j = 1, \ldots, k+1$. Clearly, $w(A^*) = w(A)$, $|A^*| = |A|$, and A^* is a heavy transversal for H^*. Because A is critical in $(H;w)$ it follows that A^* is critical in $(H^*;w^*)$. Now H^* has $k+1 \leq n$ edges and $\bigcap_{i=1}^{k+1} e_i^* = \emptyset$, so it follows that H^* is trivially n-Helly. Thus $(H^*;w^*)$ is the desired multihypergraph, and A^* is the critical set of vertices in $(H^*;w^*)$ satisfying $|A| = |A^*|$. ∎

In light of Lemma 2.1, the remainder of this paper will consider heavy transversal sets rather than multisets, and critical sets rather than critical multisets of vertices.

The *degree* of a vertex v in a hypergraph $H = (V,E)$ is $d_H(v) = \sum_{v \in e \in E} m_E(e)$. The hypergraph H is *d-regular* if $d_H(v) = d$, for all $v \in V$. A d-regular multihypergraph H is called *indecomposable* if H has no d'-regular spanning partial hypergraph with $1 \leq d' < d$. Define $D(n)$ to be the maximum d such that there exists a d-regular indecomposable hypergraph on n vertices. Huckemann and Jurkat (see [5]) were the first to prove that $D(n)$ is finite for all n. The best known upper bound, proved by Huckemann, Jurkat, and Shapley, is $D(n) \leq (n+1)^{\frac{n+1}{2}}$ (cf. [1]). Alon and Vũ [2] proved that this upper bound is asymptotically the best possible. We shall make use of this upper bound in Theorem 2.2. The following theorem is the main result of the paper.

Theorem 2.2. *There exists a function $f(n)$ such that for all weighted n-Helly hypergraphs $(H;w)$, $\tau(H^B) \leq 1$ holds for every $B \subseteq V(H)$ if and only if $\tau(H^A) \leq 1$, for every $A \subseteq V(H)$ with $|A| \leq f(n)$.*

Proof. Necessity is obvious. To prove sufficiency, assume that $\tau(H^B) \geq 2$, for some $B \subseteq V$. Let $k = \kappa(H^B)$, and consider the partial hypergraph K defined by $k+1$ hyperedges $e_1, \ldots, e_{k+1} \in E(H^B)$ with no point in common. Observe that $k < n$, because otherwise, $\tau(K) \leq 1$, by the n-Helly property of H. Let A be a heavy transversal of K. Assume that A is minimal by inclusion; that is, $H^A = K$, and furthermore, $A' \subseteq A$ and $H^{A'} = K$ implies $A' = A$. Our goal is to give an upper bound on $|A|$ that depends only on n.

Let r be the unique integer satisfying $\max\{w(A), 0\} < r \leq w(A) + 1$. Consider the subhypergraph K_A. For every $i = 1, \ldots, k+1$, choose $x_i \subseteq e_i \cap A$ such that $|x_i| = r$. The sets $x_1, x_2, \ldots, x_{k+1}$ form a hypergraph L that is r-uniform, and clearly, A is a minimal heavy transversal of L.

For a fixed $1 \le s \le r$, a set $S \subseteq A$ is called an *exact s-transversal* of L, if $|x_i \cap S| = s$, for every $i = 1, \dots, k+1$. Notice that A itself is an exact r-transversal of L. We show that L has no exact s-transversal S different from A. If $s > w(S)$, then $S = A$, by the minimality of A. If $1 \le s \le w(S) \le w(A) < r$, then $|x_i \cap (A \setminus S)| = r - s > w(A) - w(S) = w(A \setminus S)$, for every $i = 1, \dots, k+1$. Thus $A \setminus S$ is a heavy transversal of L, contradicting to the minimality of A. Therefore, L has no exact s-transversal with $1 \le s < r$.

Consider the dual hypergraph L^* with vertex set $\{x_1, \dots, x_{k+1}\}$, and with $|A|$ hyperedges: $f_j = \{x_i : a_j \in x_i\}$, for every $a_j \in A$. Because L is r-uniform, every vertex x_i, $1 \le i \le k{+}1$, is contained in exactly r hyperedges of L^*, i.e., L^* is r-regular. Furthermore, by the minimality of A, L^* has no s-regular spanning partial hypergraph with $1 \le s < r$, i.e., L^* is indecomposable. Observing that $|V(L^*)| = k + 1 \le n$ and applying the bound given by Huckemann, Jurkat, and Shapely, we obtain $r \le (n{+}1)^{\frac{n+1}{2}}$. Hence $|A| \le (k{+}1)r \le n(n{+}1)^{\frac{n+1}{2}}$, and the theorem follows by setting $f(n) = n(n+1)^{\frac{n+1}{2}}$. ∎

The proof of Theorem 2.2 relies on the fact that 2-critical, n-Helly hypergraphs have a kernel of cardinality bounded by n. This implies that, when $\tau = 2$, the number of edges in a τ-critical n-Helly hypergraph is at most n. For $\tau > 2$, this is not the case. Actually, the number of edges may be arbitrarily large as the following example shows. Let H_p be the clique hypergraph of the odd antihole $\overline{C}_{2p+1}$ (the complement of an induced cycle with $2p+1$ vertices). For every $p \ge 2$, H_p is 3-Helly, and τ-critical with $\tau = 3$. This observation indicates that the heavy transversal theorem is not true with $\tau > 1$, unless we restrict the family of n-Helly hypergraphs. Another possibility, instead of restricting n-Helly hypergraphs, is to consider 2-intersecting τ-critical hypergraphs with $\tau = t \ge 3$. It is proved in [6] that these hypergraphs have a kernel of cardinality at most t^2, called a 2-transversal, with the property that each edge meets the kernel in at least two vertices.

One reason why the proof of Theorem 2.2 works is the fact that if A is a critical, heavy transversal such that every hyperedge of some set U contains exactly d elements of A, then the dual hypergraph (U, A) is a d-regular indecomposable hypergraph. The converse is not true. Consider a triangle T with weight $1/3$ assigned to each of the three vertices of T. This weighted triangle T is indecomposable, and the vertex set is a heavy transversal. However, the vertex set is not critical because any two vertices form a heavy transversal. Therefore, determining bounds on $f(n)$ is not equivalent to the problem of determining bounds on indecomposable hypergraphs. So there remains the basic question of the order of magnitude of $f(n)$

Using the existence of large indecomposable uniform hypergraphs we can prove an exponential lower bound for $f(n)$. Actually the proof of the next theorem gives, for any n, a construction of an n-Helly weighted hypergraph with no "small" critical sets of vertices. The main ingredient of the proof is a construction – due to Füredi [4] – of 3-uniform, d-regular indecomposable hypergraphs on $n \geq 10$ vertices with $d \geq 2^{(n-6)/2}$.

Theorem 2.3. *For any $n \geq 10$, there exist weights $0 < \alpha < \beta < 1$, and an n-Helly hypergraph $(H;w)$ with $w : V \to \{\alpha, \beta\}$ such that the smallest set of vertices S satisfying $\tau(H^S) > 1$ has cardinality at least $(n/3)2^{(n-6)/2}$.*

Proof. Let (F, B) be the 3-uniform, d-regular indecomposable hypergraph on $n = |F|$ vertices constructed by Füredi [4]. In particular, $d \geq 2^{(n-6)/2}$ and $m = |B| = nd/3$. Consider the d-uniform dual hypergraph (B, F), where $F = \{f_1, \ldots, f_n\}$. To (B, F) add n vertices $a_1, \ldots, a_n$, and include each a_j to every f_i, $i \neq j$. Let $A = \{a_1, \ldots, a_n\}$ and $E = \{e_1, \ldots, e_n\}$. The resulting hypergraph $H = (A \cup B, E)$ is $(d+n-1)$-uniform, n-Helly, and 2-critical.

Define the weights $\beta = 1 - \epsilon$ and $\alpha = \frac{3}{n} - \epsilon$, where $\epsilon = \frac{1}{2mn}$. Assign weight β to the vertices of H in A and assign α to the vertices of H in B. Note that B is a heavy transversal of H, since $w(B) = m\alpha = m(3/n - \epsilon) < d = |e_i \cap B|$. Because H is 2-critical, $\tau(H^B) = \tau(H) = 2$.

We now prove that B is the only minimal subset of the vertices generating a hypergraph with $\tau > 1$. Suppose to the contrary that $S \neq B$ is a nonempty minimal subset of the vertices of H satisfying $\tau(H^S) > 1$. In particular, $Y = S \cap B$ and $X = S \cap A$ are nonempty. Also, $\{e_1, \ldots, e_n\} = H^S$, since H is 2-critical. Because (F, B) is indecomposable, there is an e_j such that $|e_j \cap Y|$ is below the average $\frac{3|Y|}{n}$, that is $|e_j \cap Y| \leq \frac{3|Y|}{n} - \frac{1}{n}$. Observe that $|Y| < m$, $|X| \leq n$, and the choice $\epsilon = 1/2mn$ imply

$$\epsilon |Y| < \frac{1}{n} - \epsilon |X|.$$

By substituting $\epsilon = 3/n - \alpha$ on the left hand side, and $\epsilon = 1 - \beta$ on the right hand side we get

$$\left(\frac{3}{n} - \alpha\right)|Y| < \frac{1}{n} - (1 - \beta)|X|,$$

or equivalently,

$$\frac{3|Y|}{n} - \frac{1}{n} + |X| < \alpha|Y| + \beta|X| = w(S).$$

Since

$$|e_j \cap S| = |e_j \cap Y| + |e_j \cap X| \leq \frac{3|Y|}{n} - \frac{1}{n} + |X|,$$

we obtain $|e_j \cap S| < w(S)$. This implies $\tau(H^S) \leq 1$, a contradiction. Thus $|S| \geq |B| = nd/3 \geq (n/3)2^{n-6/2}$ follows, concluding the proof of the theorem. ∎

In this paper we have made a connection between indecomposable hypergraphs and critical weighted n-Helly hypergraphs.

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